AP Calculus Summer Assignment

To: All students enrolled in AP Calculus AB/BC
From: AP Calculus teacher, Mr. Collins

An Introduction to Calculus:

In some ways, Calculus involves taking what you already know a step further. You know how to find the slope of a line, right? You probably don't know how to find the slope of a curve because it's constantly *changing* – but Calculus helps us do that. 'Traditional' math tells us how to find the slope of a line, and Calculus tells us how to find the slope of a curve. 'Traditional' math tells us how to find the length of a rope pulled taut, but Calculus tells us how to find the length of a curved rope. 'Traditional' math tells us how to find the area of a flat, rectangular roof, but Calculus tells us how to find the area of a curved dome-shaped roof. Get the idea?
How does Calculus manage to pull this off? Imagine a curve like this:

If you were to zoom in a few times, each part of the curve would kind of look like a line, wouldn't it? And if "a few times" wasn't enough, you could zoom in more. And more. And more. In fact, you could zoom in nearly an infinite number of times until the curve became enough like a line that you could treat it that way. "What makes calculus such a fantastic achievement is that it actually zooms in *infinitely*. In fact, everything you do in calculus involves infinity in one way or another, because if something is constantly changing, it's changing infinitely often from each infinitesimal moment to the next."
(excerpt taken from http://media.wiley.com/product_data/excerpt/84/07645249/0764524984.pdf)

This process – doing something an infinite number of times until the problem becomes figure-out-able – is the foundation of Calculus. The process is called a "limit" and it's what we'll be talking about in our first month of Calculus together.

Next year, we will be using the following resource (**Version #1 - Calculus (flippedmath.com)**).

You have two options.

You can print each Unit yourself as we go through the curriculum, or you can purchase the workbook that has all of the note-taking guides and practice sheets all together, bound in a very nice book.
We have a discount they are giving us to purchase the book. Please either purchase the book and bring it with you on the first day of school OR be prepared to print Unit 1 (62 pages) during the first week of school. AB has 8 units (382 pages), BC has 10 Units (563 pages).

FOLLOW THE CORRECT LINK!
There is a separate AP Calculus AB book and AP Calculus BC book.

Link to AB workbook: AP Calculus AB workbook (flippedmath.com)
https://www.flippedmath.com/store/p54/AP_Calculus_AB_Workbook.html#/
Coupon code: **Steinbrenner25AB**

Link to BC workbook: AP Calculus BC workbook (flippedmath.com)
https://www.flippedmath.com/store/p66/AP_Calculus_BC_Workbook.html#/
Coupon code: **Steinbrenner25BC**

The following is the Summer Assignment for students who are taking AP Calculus at Steinbrenner HS for the 2022-2023 school year. If you took AP Calculus AB last year, this is a new assignment, and ***DOES*** need to be completed by August 10th, 2022, and brought to class on the first day of school.

Summer practice problem directions:

1. Read the following algebra review pages 1-11. You may print or take notes if you find it helpful.
2. On your own paper, **copy and complete** the following exercises (1-108, 127-144, 149-156), ***showing ALL work***! ***Write your name on the top of each page you use.*** You may use the front and back of each page. If you only have answers, the grade will be a zero. ALL problems require you to show your work. All problem types have examples in pages 1-11. If you get stuck on any problems, consult the examples, consult fellow classmates, consult other resources.

Please adhere to these directions or the summer assignment grade may be a zero.

REVIEW OF ALGEBRA

Here we review the basic rules and procedures of algebra that you need to know in order to be successful in calculus.

ARITHMETIC OPERATIONS

The real numbers have the following properties:

$$\begin{array}{llll} a + b = b + a & ab = ba & & \text{(Commutative Law)} \\ (a + b) + c = a + (b + c) & & (ab)c = a(bc) & \text{(Associative Law)} \\ a(b + c) = ab + ac & & & \text{(Distributive law)} \end{array}$$

In particular, putting $a = -1$ in the Distributive Law, we get

$$-(b + c) = (-1)(b + c) = (-1)b + (-1)c$$

and so

$$\boxed{-(b + c) = -b - c}$$

EXAMPLE 1

(a) $(3xy)(-4x) = 3(-4)x^2y = -12x^2y$

(b) $2t(7x + 2tx - 11) = 14tx + 4t^2x - 22t$

(c) $4 - 3(x - 2) = 4 - 3x + 6 = 10 - 3x$ ■

If we use the Distributive Law three times, we get

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$$

This says that we multiply two factors by multiplying each term in one factor by each term in the other factor and adding the products. Schematically, we have

$$(a + b)(c + d)$$

In the case where $c = a$ and $d = b$, we have

$$(a + b)^2 = a^2 + ba + ab + b^2$$

or

$$\boxed{(a + b)^2 = a^2 + 2ab + b^2} \tag{1}$$

Similarly, we obtain

$$\boxed{(a - b)^2 = a^2 - 2ab + b^2} \tag{2}$$

EXAMPLE 2

(a) $(2x + 1)(3x - 5) = 6x^2 + 3x - 10x - 5 = 6x^2 - 7x - 5$

(b) $(x + 6)^2 = x^2 + 12x + 36$

(c) $3(x - 1)(4x + 3) - 2(x + 6) = 3(4x^2 - x - 3) - 2x - 12$

$\qquad = 12x^2 - 3x - 9 - 2x - 12 = 12x^2 - 5x - 21$ ■

FRACTIONS

To add two fractions with the same denominator, we use the Distributive Law:

$$\frac{a}{b} + \frac{c}{b} = \frac{1}{b} \times a + \frac{1}{b} \times c = \frac{1}{b}(a + c) = \frac{a + c}{b}$$

Thus, it is true that

$$\boxed{\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b}}$$

But remember to avoid the following common error:

$$\frac{a}{b + c} \neq \frac{a}{b} + \frac{a}{c}$$

(For instance, take $a = b = c = 1$ to see the error.)

To add two fractions with different denominators, we use a common denominator:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}}$$

We multiply such fractions as follows:

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}}$$

In particular, it is true that

$$\boxed{\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}}$$

To divide two fractions, we invert and multiply:

$$\boxed{\frac{\dfrac{a}{b}}{\dfrac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}}$$

EXAMPLE 3

(a) $\dfrac{x + 3}{x} = \dfrac{x}{x} + \dfrac{3}{x} = 1 + \dfrac{3}{x}$

(b) $\dfrac{3}{x - 1} + \dfrac{x}{x + 2} = \dfrac{3(x + 2) + x(x - 1)}{(x - 1)(x + 2)} = \dfrac{3x + 6 + x^2 - x}{x^2 + x - 2} = \dfrac{x^2 + 2x + 6}{x^2 + x - 2}$

(c) $\dfrac{s^2t}{u} \cdot \dfrac{ut}{-2} = \dfrac{s^2t^2u}{-2u} = -\dfrac{s^2t^2}{2}$

(d) $$\frac{\frac{x}{y}+1}{1-\frac{y}{x}} = \frac{\frac{x+y}{y}}{\frac{x-y}{x}} = \frac{x+y}{y} \times \frac{x}{x-y} = \frac{x(x+y)}{y(x-y)} = \frac{x^2+xy}{xy-y^2}$$ ■

FACTORING

We have used the Distributive Law to expand certain algebraic expressions. We sometimes need to reverse this process (again using the Distributive Law) by factoring an expression as a product of simpler ones. The easiest situation occurs when the expression has a common factor as follows:

$$3x(x-2) \underset{\longleftarrow \text{ Factoring}}{\overset{\text{Expanding} \longrightarrow}{=}} 3x^2 - 6x$$

To factor a quadratic of the form $x^2 + bx + c$ we note that

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

EXAMPLE 4 Factor $x^2 + 5x - 24$.

SOLUTION The two integers that add to give 5 and multiply to give -24 are -3 and 8. Therefore

$$x^2 + 5x - 24 = (x - 3)(x + 8)$$ ■

EXAMPLE 5 Factor $2x^2 - 7x - 4$.

SOLUTION Even though the coefficient of x^2 is not 1, we can still look for factors of the form $2x + r$ and $x + s$, where $rs = -4$. Experimentation reveals that

$$2x^2 - 7x - 4 = (2x + 1)(x - 4)$$ ■

Some special quadratics can be factored by using Equations 1 or 2 (from right to left) or by using the formula for a difference of squares:

3 $$\boxed{a^2 - b^2 = (a - b)(a + b)}$$

The analogous formula for a difference of cubes is

4 $$\boxed{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

which you can verify by expanding the right side. For a sum of cubes we have

5 $$\boxed{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

EXAMPLE 6

(a) $x^2 - 6x + 9 = (x - 3)^2$ (Equation 2; $a = x$, $b = 3$)

(b) $4x^2 - 25 = (2x - 5)(2x + 5)$ (Equation 3; $a = 2x$, $b = 5$)

(c) $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$ (Equation 5; $a = x$, $b = 2$) ■

EXAMPLE 7 Simplify $\dfrac{x^2 - 16}{x^2 - 2x - 8}$.

SOLUTION Factoring numerator and denominator, we have

$$\frac{x^2 - 16}{x^2 - 2x - 8} = \frac{(x-4)(x+4)}{(x-4)(x+2)} = \frac{x+4}{x+2}$$

To factor polynomials of degree 3 or more, we sometimes use the following fact.

> **6 The Factor Theorem** If P is a polynomial and $P(b) = 0$, then $x - b$ is a factor of $P(x)$.

EXAMPLE 8 Factor $x^3 - 3x^2 - 10x + 24$.

SOLUTION Let $P(x) = x^3 - 3x^2 - 10x + 24$. If $P(b) = 0$, where b is an integer, then b is a factor of 24. Thus, the possibilities for b are ±1, ±2, ±3, ±4, ±6, ±8, ±12, and ±24. We find that $P(1) = 12$, $P(-1) = 30$, $P(2) = 0$. By the Factor Theorem, $x - 2$ is a factor. Instead of substituting further, we use long division as follows:

$$\begin{array}{r|l}
 & x^2 - x - 12 \\
x-2 & x^3 - 3x^2 - 10x + 24 \\
 & x^3 - 2x^2 \\
\hline
 & \quad -x^2 - 10x \\
 & \quad -x^2 + 2x \\
\hline
 & \qquad -12x + 24 \\
 & \qquad -12x + 24 \\
\hline
\end{array}$$

Therefore
$$\begin{aligned} x^3 - 3x^2 - 10x + 24 &= (x-2)(x^2 - x - 12) \\ &= (x-2)(x+3)(x-4) \end{aligned}$$

COMPLETING THE SQUARE

Completing the square is a useful technique for graphing parabolas or integrating rational functions. Completing the square means rewriting a quadratic $ax^2 + bx + c$ in the form $a(x+p)^2 + q$ and can be accomplished by:

1. Factoring the number a from the terms involving x.
2. Adding and subtracting the square of half the coefficient of x.

In general, we have

$$\begin{aligned} ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x\right] + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \end{aligned}$$

EXAMPLE 9 Rewrite $x^2 + x + 1$ by completing the square.

SOLUTION The square of half the coefficient of x is $\frac{1}{4}$. Thus

$$x^2 + x + 1 = x^2 + x + \tfrac{1}{4} - \tfrac{1}{4} + 1 = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$$

EXAMPLE 10

$$2x^2 - 12x + 11 = 2[x^2 - 6x] + 11 = 2[x^2 - 6x + 9 - 9] + 11$$
$$= 2[(x-3)^2 - 9] + 11 = 2(x-3)^2 - 7$$ ■

QUADRATIC FORMULA

By completing the square as above we can obtain the following formula for the roots of a quadratic equation.

7 The Quadratic Formula The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 11 Solve the equation $5x^2 + 3x - 3 = 0$.

SOLUTION With $a = 5$, $b = 3$, $c = -3$, the quadratic formula gives the solutions

$$x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-3)}}{2(5)} = \frac{-3 \pm \sqrt{69}}{10}$$ ■

The quantity $b^2 - 4ac$ that appears in the quadratic formula is called the **discriminant**. There are three possibilities:

1. If $b^2 - 4ac > 0$, the equation has two real roots.
2. If $b^2 - 4ac = 0$, the roots are equal.
3. If $b^2 - 4ac < 0$, the equation has no real root. (The roots are complex.)

These three cases correspond to the fact that the number of times the parabola $y = ax^2 + bx + c$ crosses the x-axis is 2, 1, or 0 (see Figure 1). In case (3) the quadratic $ax^2 + bx + c$ can't be factored and is called **irreducible.**

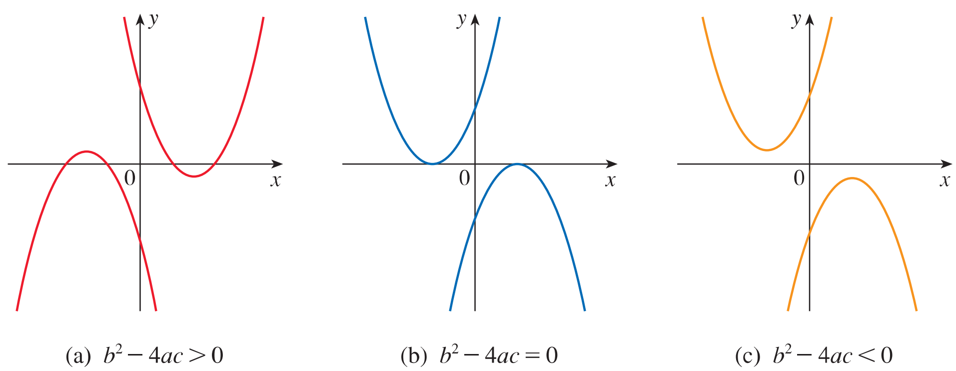

FIGURE 1
Possible graphs of $y = ax^2 + bx + c$

EXAMPLE 12 The quadratic $x^2 + x + 2$ is irreducible because its discriminant is negative:

$$b^2 - 4ac = 1^2 - 4(1)(2) = -7 < 0$$

Therefore, it is impossible to factor $x^2 + x + 2$. ■

THE BINOMIAL THEOREM

Recall the binomial expression from Equation 1:

$$(a+b)^2 = a^2 + 2ab + b^2$$

If we multiply both sides by $(a+b)$ and simplify, we get the binomial expansion

8
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Repeating this procedure, we get

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In general, we have the following formula.

9 **The Binomial Theorem** If k is a positive integer, then

$$\begin{aligned}(a+b)^k = a^k + ka^{k-1}b &+ \frac{k(k-1)}{1\cdot 2}a^{k-2}b^2 \\ &+ \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3 \\ &+ \cdots + \frac{k(k-1)\cdots(k-n+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot n}a^{k-n}b^n \\ &+ \cdots + kab^{k-1} + b^k\end{aligned}$$

EXAMPLE 13 Expand $(x-2)^5$.

SOLUTION Using the Binomial Theorem with $a = x$, $b = -2$, $k = 5$, we have

$$\begin{aligned}(x-2)^5 &= x^5 + 5x^4(-2) + \frac{5\cdot 4}{1\cdot 2}x^3(-2)^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\end{aligned}$$

■

RADICALS

The most commonly occurring radicals are square roots. The symbol $\sqrt{}$ means "the positive square root of." Thus

$$x = \sqrt{a} \qquad \text{means} \qquad x^2 = a \quad \text{and} \quad x \geqslant 0$$

Since $a = x^2 \geqslant 0$, the symbol $\sqrt{a}$ makes sense only when $a \geqslant 0$. Here are two rules for working with square roots:

10
$$\sqrt{ab} = \sqrt{a}\,\sqrt{b} \qquad\qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

However, there is no similar rule for the square root of a sum. In fact, you should remember to avoid the following common error:

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

(For instance, take $a = 9$ and $b = 16$ to see the error.)

EXAMPLE 14

(a) $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$ (b) $\sqrt{x^2 y} = \sqrt{x^2}\sqrt{y} = |x|\sqrt{y}$

Notice that $\sqrt{x^2} = |x|$ because $\sqrt{\ }$ indicates the positive square root. (See **Absolute Value**.) ■

In general, if n is a positive integer,

$$x = \sqrt[n]{a} \qquad \text{means} \qquad x^n = a$$

If n is even, then $a \geqslant 0$ and $x \geqslant 0$.

Thus $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$, but $\sqrt[4]{-8}$ and $\sqrt[6]{-8}$ are not defined. The following rules are valid:

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad\qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

EXAMPLE 15 $\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = \sqrt[3]{x^3}\,\sqrt[3]{x} = x\sqrt[3]{x}$ ■

To **rationalize** a numerator or denominator that contains an expression such as $\sqrt{a} - \sqrt{b}$, we multiply both the numerator and the denominator by the conjugate radical $\sqrt{a} + \sqrt{b}$. Then we can take advantage of the formula for a difference of squares:

$$\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$$

EXAMPLE 16 Rationalize the numerator in the expression $\dfrac{\sqrt{x+4} - 2}{x}$.

SOLUTION We multiply the numerator and the denominator by the conjugate radical $\sqrt{x+4} + 2$:

$$\frac{\sqrt{x+4} - 2}{x} = \left(\frac{\sqrt{x+4} - 2}{x}\right)\left(\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}\right) = \frac{(x+4) - 4}{x\left(\sqrt{x+4} + 2\right)}$$

$$= \frac{x}{x\left(\sqrt{x+4} + 2\right)} = \frac{1}{\sqrt{x+4} + 2}$$ ■

EXPONENTS

Let a be any positive number and let n be a positive integer. Then, by definition,

1. $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$
2. $a^0 = 1$
3. $a^{-n} = \dfrac{1}{a^n}$
4. $a^{1/n} = \sqrt[n]{a}$

   $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ $\qquad m$ is any integer

11 Laws of Exponents Let a and b be positive numbers and let r and s be any rational numbers (that is, ratios of integers). Then

1. $a^r \times a^s = a^{r+s}$ **2.** $\dfrac{a^r}{a^s} = a^{r-s}$ **3.** $(a^r)^s = a^{rs}$

4. $(ab)^r = a^r b^r$ **5.** $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r} \quad b \neq 0$

In words, these five laws can be stated as follows:

1. To multiply two powers of the same number, we add the exponents.
2. To divide two powers of the same number, we subtract the exponents.
3. To raise a power to a new power, we multiply the exponents.
4. To raise a product to a power, we raise each factor to the power.
5. To raise a quotient to a power, we raise both numerator and denominator to the power.

EXAMPLE 17

(a) $2^8 \times 8^2 = 2^8 \times (2^3)^2 = 2^8 \times 2^6 = 2^{14}$

(b) $$\frac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}} = \frac{\dfrac{1}{x^2} - \dfrac{1}{y^2}}{\dfrac{1}{x} + \dfrac{1}{y}} = \frac{\dfrac{y^2 - x^2}{x^2y^2}}{\dfrac{y + x}{xy}} = \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y + x}$$

$$= \frac{(y - x)(y + x)}{xy(y + x)} = \frac{y - x}{xy}$$

(c) $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$ Alternative solution: $4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

(d) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

(e) $\left(\dfrac{x}{y}\right)^3 \left(\dfrac{y^2x}{z}\right)^4 = \dfrac{x^3}{y^3} \cdot \dfrac{y^8x^4}{z^4} = x^7y^5z^{-4}$ ■

INEQUALITIES

When working with inequalities, note the following rules.

Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that *if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality.* For example, if we take the inequality

$3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 18 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set*.

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\tfrac{4}{6} = -\tfrac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 19 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

■■ A visual method for solving Example 19 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 2) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

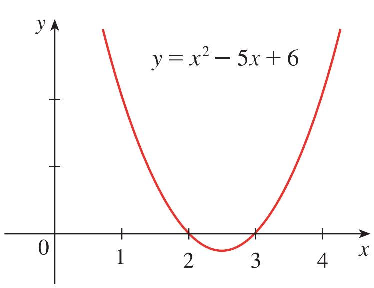

FIGURE 2

Another method for obtaining the information in the chart is to use *test values*. For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus, the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

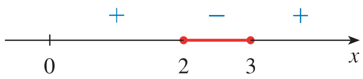

FIGURE 3

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 3. ■

EXAMPLE 20 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x-1)(x+4) > 0$$

As in Example 19 we solve the corresponding equation $x(x-1)(x+4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign as shown in the following chart.

Interval	x	$x-1$	$x+4$	$x(x-1)(x+4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

FIGURE 4

The solution is illustrated in Figure 4. ■

ABSOLUTE VALUE

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0$$

$$|\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

12

$$\boxed{\begin{aligned} |a| &= a && \text{if } a \geqslant 0 \\ |a| &= -a && \text{if } a < 0 \end{aligned}}$$

■■ Remember that if a is negative, then $-a$ is positive.

EXAMPLE 21 Express $|3x - 2|$ without using the absolute-value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

■

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus, $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore, the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (12), we then have the equation

13
$$\boxed{\sqrt{a^2} = |a|}$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

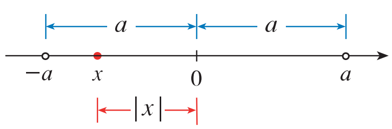

FIGURE 5

Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

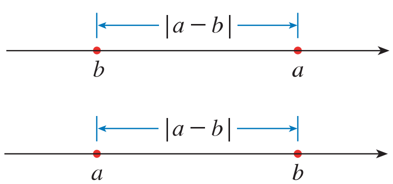

FIGURE 6

Length of a line segment $= |a - b|$

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 5 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 6.)

EXAMPLE 22 Solve $|2x - 5| = 3$.

SOLUTION By Property 4 of absolute values, $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \qquad \text{or} \qquad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus, $x = 4$ or $x = 1$. ■

EXAMPLE 23 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of absolute values, $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval $(3, 7)$.

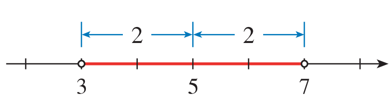

FIGURE 7

SOLUTION 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 7 we see that this is the interval $(3, 7)$. ■

EXAMPLE 24 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of absolute values, $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \qquad \text{or} \qquad 3x + 2 \leqslant -4$$

In the first case, $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case, $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

EXERCISES

A Click here for answers.

S Click here for solutions.

1–16 ■ Expand and simplify.

1. $(-6ab)(0.5ac)$

2. $-(2x^2y)(-xy^4)$

3. $2x(x - 5)$

4. $(4 - 3x)x$

5. $-2(4 - 3a)$

6. $8 - (4 + x)$

7. $4(x^2 - x + 2) - 5(x^2 - 2x + 1)$

8. $5(3t - 4) - (t^2 + 2) - 2t(t - 3)$

9. $(4x - 1)(3x + 7)$

10. $x(x - 1)(x + 2)$

11. $(2x - 1)^2$

12. $(2 + 3x)^2$

13. $y^4(6 - y)(5 + y)$

14. $(t - 5)^2 - 2(t + 3)(8t - 1)$

15. $(1 + 2x)(x^2 - 3x + 1)$

16. $(1 + x - x^2)^2$

17–28 ■ Perform the indicated operations and simplify.

17. $\dfrac{2 + 8x}{2}$

18. $\dfrac{9b - 6}{3b}$

19. $\dfrac{1}{x + 5} + \dfrac{2}{x - 3}$

20. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1}$

21. $u + 1 + \dfrac{u}{u + 1}$

22. $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{4}{b^2}$

23. $\dfrac{x/y}{z}$

24. $\dfrac{x}{y/z}$

25. $\left(\dfrac{-2r}{s}\right)\left(\dfrac{s^2}{-6t}\right)$

26. $\dfrac{a}{bc} \div \dfrac{b}{ac}$

27. $\dfrac{1 + \dfrac{1}{c - 1}}{1 - \dfrac{1}{c - 1}}$

28. $1 + \dfrac{1}{1 + \dfrac{1}{1 + x}}$

29–48 ■ Factor the expression.

29. $2x + 12x^3$

30. $5ab - 8abc$

31. $x^2 + 7x + 6$

32. $x^2 - x - 6$

33. $x^2 - 2x - 8$

34. $2x^2 + 7x - 4$

35. $9x^2 - 36$

36. $8x^2 + 10x + 3$

37. $6x^2 - 5x - 6$

38. $x^2 + 10x + 25$

39. $t^3 + 1$

40. $4t^2 - 9s^2$

41. $4t^2 - 12t + 9$

42. $x^3 - 27$

43. $x^3 + 2x^2 + x$

44. $x^3 - 4x^2 + 5x - 2$

45. $x^3 + 3x^2 - x - 3$

46. $x^3 - 2x^2 - 23x + 60$

47. $x^3 + 5x^2 - 2x - 24$

48. $x^3 - 3x^2 - 4x + 12$

49–54 ■ Simplify the expression.

49. $\dfrac{x^2 + x - 2}{x^2 - 3x + 2}$

50. $\dfrac{2x^2 - 3x - 2}{x^2 - 4}$

51. $\dfrac{x^2 - 1}{x^2 - 9x + 8}$

52. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - x - 12}$

53. $\dfrac{1}{x + 3} + \dfrac{1}{x^2 - 9}$

54. $\dfrac{x}{x^2 + x - 2} - \dfrac{2}{x^2 - 5x + 4}$

55–60 ■ Complete the square.

55. $x^2 + 2x + 5$

56. $x^2 - 16x + 80$

57. $x^2 - 5x + 10$

58. $x^2 + 3x + 1$

59. $4x^2 + 4x - 2$

60. $3x^2 - 24x + 50$

61–68 ■ Solve the equation.

61. $x^2 + 9x - 10 = 0$

62. $x^2 - 2x - 8 = 0$

63. $x^2 + 9x - 1 = 0$

64. $x^2 - 2x - 7 = 0$

65. $3x^2 + 5x + 1 = 0$

66. $2x^2 + 7x + 2 = 0$

67. $x^3 - 2x + 1 = 0$

68. $x^3 + 3x^2 + x - 1 = 0$

69–72 ■ Which of the quadratics are irreducible?

69. $2x^2 + 3x + 4$

70. $2x^2 + 9x + 4$

71. $3x^2 + x - 6$

72. $x^2 + 3x + 6$

73–76 ■ Use the Binomial Theorem to expand the expression.

73. $(a + b)^6$

74. $(a + b)^7$

75. $(x^2 - 1)^4$

76. $(3 + x^2)^5$

77–82 ■ Simplify the radicals.

77. $\sqrt{32}\sqrt{2}$ **78.** $\dfrac{\sqrt[3]{-2}}{\sqrt[3]{54}}$ **79.** $\dfrac{\sqrt[4]{32x^4}}{\sqrt[4]{2}}$

80. $\sqrt{xy}\sqrt{x^3y}$ **81.** $\sqrt{16a^4b^3}$ **82.** $\dfrac{\sqrt[5]{96a^6}}{\sqrt[5]{3a}}$

83–100 ■ Use the Laws of Exponents to rewrite and simplify the expression.

83. $3^{10} \times 9^8$ **84.** $2^{16} \times 4^{10} \times 16^6$

85. $\dfrac{x^9(2x)^4}{x^3}$ **86.** $\dfrac{a^n \times a^{2n+1}}{a^{n-2}}$

87. $\dfrac{a^{-3}b^4}{a^{-5}b^5}$ **88.** $\dfrac{x^{-1}+y^{-1}}{(x+y)^{-1}}$

89. $3^{-1/2}$ **90.** $96^{1/5}$

91. $125^{2/3}$ **92.** $64^{-4/3}$

93. $(2x^2y^4)^{3/2}$ **94.** $(x^{-5}y^3z^{10})^{-3/5}$

95. $\sqrt[5]{y^6}$ **96.** $(\sqrt[4]{a})^3$

97. $\dfrac{1}{(\sqrt{t})^5}$ **98.** $\dfrac{\sqrt[8]{x^5}}{\sqrt[4]{x^3}}$

99. $\sqrt[4]{\dfrac{t^{1/2}\sqrt{st}}{s^{2/3}}}$ **100.** $\sqrt[4]{r^{2n+1}} \times \sqrt[4]{r^{-1}}$

101–108 ■ Rationalize the expression.

101. $\dfrac{\sqrt{x}-3}{x-9}$ **102.** $\dfrac{(1/\sqrt{x})-1}{x-1}$

103. $\dfrac{x\sqrt{x}-8}{x-4}$ **104.** $\dfrac{\sqrt{2+h}+\sqrt{2-h}}{h}$

105. $\dfrac{2}{3-\sqrt{5}}$ **106.** $\dfrac{1}{\sqrt{x}-\sqrt{y}}$

107. $\sqrt{x^2+3x+4}-x$ **108.** $\sqrt{x^2+x}-\sqrt{x^2-x}$

109–116 ■ State whether or not the equation is true for all values of the variable.

109. $\sqrt{x^2} = x$ **110.** $\sqrt{x^2+4} = |x| + 2$

111. $\dfrac{16+a}{16} = 1 + \dfrac{a}{16}$ **112.** $\dfrac{1}{x^{-1}+y^{-1}} = x + y$

113. $\dfrac{x}{x+y} = \dfrac{1}{1+y}$ **114.** $\dfrac{2}{4+x} = \dfrac{1}{2} + \dfrac{2}{x}$

115. $(x^3)^4 = x^7$

116. $6 - 4(x+a) = 6 - 4x - 4a$

117–126 ■ Rewrite the expression without using the absolute value symbol.

117. $|5-23|$ **118.** $|\pi - 2|$

119. $|\sqrt{5}-5|$ **120.** $||-2|-|-3||$

121. $|x-2|$ if $x < 2$ **122.** $|x-2|$ if $x > 2$

123. $|x+1|$ **124.** $|2x-1|$

125. $|x^2+1|$ **126.** $|1-2x^2|$

127–142 ■ Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

127. $2x + 7 > 3$ **128.** $4 - 3x \geqslant 6$

129. $1 - x \leqslant 2$ **130.** $1 + 5x > 5 - 3x$

131. $0 \leqslant 1 - x < 1$ **132.** $1 < 3x + 4 \leqslant 16$

133. $(x-1)(x-2) > 0$ **134.** $x^2 < 2x + 8$

135. $x^2 < 3$ **136.** $x^2 \geqslant 5$

137. $x^3 - x^2 \leqslant 0$

138 $(x+1)(x-2)(x+3) \geqslant 0$

139. $x^3 > x$ **140.** $x^3 + 3x < 4x^2$

141. $\dfrac{1}{x} < 4$ **142.** $-3 < \dfrac{1}{x} \leqslant 1$

143. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

144. Use the relationship between C and F given in Exercise 143 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

145. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.
(a) If the ground temperature is 20°C, write a formula for the temperature at height h.
(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

146. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

147–148 ■ Solve the equation for x.

147. $|x+3| = |2x+1|$ **148.** $|3x+5| = 1$

149–156 ■ Solve the inequality.

149. $|x| < 3$ **150.** $|x| \geqslant 3$

151. $|x-4| < 1$ **152.** $|x-6| < 0.1$

153. $|x+5| \geqslant 2$ **154.** $|x+1| \geqslant 3$

155. $|2x-3| \leqslant 0.4$ **156.** $|5x-2| < 6$

157. Solve the inequality $a(bx - c) \geqslant bc$ for x, assuming that a, b, and c are positive constants.

158. Solve the inequality $ax + b < c$ for x, assuming that a, b, and c are negative constants.

159 Prove that $|ab| = |a||b|$. [*Hint:* Use Equation 3.]

160. Show that if $0 < a < b$, then $a^2 < b^2$.